

Dispersion properties of transverse waves propagating in the electrically polarized BEC

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Further development of the method of the quantum hydrodynamics in application to a Bose-Einstein condensate is presented. To consider evolution of polarization direction along with particles movement we have developed the set of quantum hydrodynamics equations, which includes equation of the polarization evolution and the polarization current evolution along with the continuity equation and the Euler equation (momentum balance equation). Dispersion properties of transverse waves including the electromagnetic waves propagating through the BEC are considered. For this purpose we consider whole system of the Maxwell's equations for field dynamic description. This approximation gives us possibility to consider electromagnetic waves along with matter waves. We show that in the electrically polarized BEC we have a splitting of the electromagnetic waves on two branches. We have four solution, two for the electromagnetic waves and two for the matter waves, the last two are the concentration-polarization waves. In this approximation we find anisotropy in dispersion dependencies for the all four waves. Whereas in the longitudinal approximation we have two matter waves only which show no anisotropy.

I. INTRODUCTION

At studying of an electrically polarized Bose-Einstein condensate (BEC) the generalization of the Gross-Pitaevskii (GP) equation [1]- [5] and some it's further generalization [6], [7], see also review papers [8]- [11], especially we want to admit recent review written by M. A. Baranov et. al. [12]. This nonlinear Schrodinger equation defines the complex scalar wave function $\psi(\mathbf{r}, t)$, which, as in the case of the non-polarized BEC, describes evolution of concentration $n(\mathbf{r}, t) = \psi^*(\mathbf{r}, t)\psi(\mathbf{r}, t)$. When direction of polarization of each particle \mathbf{d} is not changing during the considering process and all of them have same direction we can write formula for density of polarization $\mathbf{P} = \psi^*(\mathbf{r}, t)\mathbf{d}\psi(\mathbf{r}, t)$. Thus, polarization density changes due to particles movement, and, consequently, changing of concentration. The waves of polarization can easily propagate even in the ferroelectrics as well as spin waves in ferromagnetics. All the more we can expect existence of them in polarized BEC. To consider evolution of polarization direction along with particles movement we have developed the set of quantum hydrodynamics equations, which includes equation of the polarization evolution and the polarization current evolution along with the continuity equation and the Euler equation (the momentum balance equation) [13]- [16].

It was shown in Ref. [13] that Hamiltonian of the electric dipole interaction correct form which corresponds to the Maxwell's equation is

$$H_{dd} = -\partial^\alpha \partial^\beta \frac{1}{r} \cdot d_1^\alpha d_2^\beta. \quad (1)$$

Using well-known identity

$$-\partial^\alpha \partial^\beta \frac{1}{r} = \frac{\delta^{\alpha\beta} - 3r^\alpha r^\beta / r^2}{r^3} + \frac{4\pi}{3} \delta^{\alpha\beta} \delta(\mathbf{r}), \quad (2)$$

we can see that the Hamiltonian (1) differs from usually used one

$$H_{dd} = \frac{\delta^{\alpha\beta} - 3r^\alpha r^\beta / r^2}{r^3} d_1^\alpha d_2^\beta. \quad (3)$$

Necessity to consider the Hamiltonian of electric dipole interaction in the form (1) caused by the fact that it must accord to the Maxwell's equation. This connection exists since Maxwell's equations describe electric field and it's connection with the sources. In our case source is the density of electric polarization. Action of the electric field on the density of polarization come in to equations of motion via the force field. As the result it describes interaction of polarization (electric dipole moments).

It has been shown in Ref.s [17]- [19] that using of the non-linear Schrodinger equation which generalized the GP equation for the system of the electrically polarized BEC and using of the formula (3) for description of the dipole-dipole interaction lead to the appearance of the contribution of the equilibrium polarization in the dispersion dependence of the Bogoliubov's mode, this contribution leads to the anisotropy of the dispersion dependence. In contrast with it, in our previous papers [13]- [16], we have shown that at account of the Hamiltonian (1) and consideration of evolution of electric dipole directions we have got dispersion dependencies for two waves, that means what account of polarization evolution leads to appearing of new wave in the BEC. Moreover, we have found that these dispersion dependencies are isotropic, and contain no dependence on direction between direction of the wave propagation and direction of the equilibrium polarization.

In this paper we suggest more general model in comparison with our recent papers. We consider whole system of the Maxwell's equations instead of the pair quasi-static equations describing the electric field considered earlier. This approximation gives us possibility to consider electromagnetic waves along with matter waves. Below we

will show that in the electrically polarized BEC we have a splitting of the electromagnetic waves on two branches. So, we have four solution, two for the electromagnetic waves and two for the matter waves, the last two are waves of concentration-polarization. It is important and interestingly to admit that in this approximation we find anisotropy in the dispersion dependencies for all four waves.

This paper is organized as follows. In Sec. II we present the quantum hydrodynamics model for description of the electrically polarized BEC. In Sec. III we present dispersion equation and describe the method to get it. In Sec. IV we consider important consequence of the obtained dispersion equation. In Sec. V brief discussion of obtained results is presented.

II. BASIC EQUATIONS

In our previous paper [13] we have presented the detailed derivation of the QHD equation for the electrically polarized BEC. These equations were directly derived from the microscopic many-particle Schrodinger equation. Let us briefly present the set of the quantum hydrodynamics equations.

The first equation of the QHD equations system is the continuity equation

$$\partial_t n + \partial^\alpha (n v^\alpha) = 0. \quad (4)$$

The momentum balance equation for the polarized BEC has the form

$$\begin{aligned} & mn(\partial_t + \mathbf{v} \cdot \nabla) v^\alpha + \partial_\beta p^{\alpha\beta} \\ & - \frac{\hbar^2}{4m} \partial^\alpha \Delta n + \frac{\hbar^2}{4m} \partial^\beta \left(\frac{\partial^\alpha n \cdot \partial^\beta n}{n} \right) \\ & = \Upsilon n \partial^\alpha n + \frac{1}{2} \Upsilon_2 \partial^\alpha \Delta n^2 + P^\beta \partial^\alpha E^\beta, \end{aligned} \quad (5)$$

where

$$\Upsilon = \frac{4\pi}{3} \int dr(r)^3 \frac{\partial U(r)}{\partial r}, \quad (6)$$

and

$$\Upsilon_2 \equiv \frac{\pi}{30} \int dr(r)^5 \frac{\partial U(r)}{\partial r}, \quad (7)$$

are the numerical coefficients. In equation (5) we define a parameter Υ_2 as (7). This definition differs from the one in the work [20]. Terms proportional to \hbar^2 appear as a result of usage of the quantum kinematics. The first two members at the right side of the equation (5) are first terms of expansion of the quantum stress tensor. They occur because of taking into account of the SRI potential U_{ij} . The interaction potential U_{ij} determinates the macroscopic interaction constants Υ and Υ_2 . The last two members of the equation (5) describe force fields that affect the dipole moment

in a unit of volume as the effect of the external electrical field and the field produced by other dipoles, respectively. The last member is written using the self-consistent field approximation [16], [21]. $p^{\alpha\beta}(\mathbf{r}, t)$ is the tensor of the kinetic pressure, which depends on particle thermal velocities and makes no contribution into the BEC dynamics at temperatures near zero.

The first order by the interaction radius constant for dilute gases has the form

$$\Upsilon = -\frac{4\pi\hbar^2 a}{m}, \quad (8)$$

where a is the scattering length SL [20, 22].

We have also the field equations

$$\nabla \mathbf{E}(\mathbf{r}, t) = -4\pi \nabla \mathbf{P}(\mathbf{r}, t), \quad (9)$$

and

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = 0. \quad (10)$$

In the case particles does not contain the dipole moment, the continuity equation and the momentum balance equation form a closed system of equations. The new physical value emerges, when the dipole moment is taken into account in the momentum balance equation, it is the polarization vector field $P^\alpha(\mathbf{r}, t)$. This causes system of equations to become incomplete.

Next equation we need for investigation of the collective excitation dispersion is the equation of polarization evolution

$$\partial_t P^\alpha(\mathbf{r}, t) + \partial^\beta R^{\alpha\beta}(\mathbf{r}, t) = 0, \quad (11)$$

$R^{\alpha\beta}(\mathbf{r}, t)$ is the current of polarization.

The equation (11) does not contain information about the effect of the interaction on the polarization evolution. The evolution equation of $R^{\alpha\beta}(\mathbf{r}, t)$ can be constructed by analogy with the above derived equations. Using the self-consistent field approximation of the dipole-dipole interaction we obtain the equation for the polarization current $R^{\alpha\beta}(\mathbf{r}, t)$ evolution

$$\begin{aligned} & \partial_t R^{\alpha\beta} + \partial^\gamma \left(R^{\alpha\beta} v^\gamma + R^{\alpha\gamma} v^\beta - P^\alpha v^\beta v^\gamma \right) \\ & + \frac{1}{m} \partial^\gamma r^{\alpha\beta\gamma} - \frac{\hbar^2}{4m^2} \partial_\beta \Delta P^\alpha \\ & + \frac{\hbar^2}{8m^2} \partial^\gamma \left(\frac{\partial_\beta P^\alpha \partial_\gamma n}{n} + \frac{\partial_\gamma P^\alpha \partial_\beta n}{n} \right) \\ & = \frac{1}{m} \Upsilon \partial^\beta \left(n P^\alpha \right) + \frac{\sigma}{m} \frac{P^\alpha P^\gamma}{n} \partial^\beta E^\gamma. \end{aligned} \quad (12)$$

Here $r^{\alpha\beta\gamma}(\mathbf{r}, t)$ presents the contribution of the thermal movement of the polarized particles into the dynamics of $R^{\alpha\beta}(\mathbf{r}, t)$. As we deal with the BEC below, the contribution of $r^{\alpha\beta\gamma}(\mathbf{r}, t)$ may be neglected. The last term of the formula (12) includes both external electrical field and the

self-consistent field that particle dipoles create. This term contain numerical constant σ .

Equations (9) and (10) emerge in the considered non-relativistic limit and allow to study longitudinal waves only. When we use term longitudinal we suggest that direction of the electric field perturbation is parallel to the direction of the wave propagation. This case has been considered in Ref.s [13]- [15]. It is well-known that equations (9) and (10) are the part of the set of the Maxwell's equations. Therefore, if we want to consider the transverse waves we should use the whole set of the Maxwell's equations, which has well-known form

$$\nabla \mathbf{B}(\mathbf{r}, t) = 0, \quad (13)$$

$$\nabla \mathbf{E}(\mathbf{r}, t) = -4\pi \nabla \mathbf{P}(\mathbf{r}, t), \quad (14)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{1}{c} \partial_t \mathbf{B}(\mathbf{r}, t), \quad (15)$$

and

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \partial_t \mathbf{E}(\mathbf{r}, t) + \frac{4\pi}{c} \partial_t \mathbf{P}(\mathbf{r}, t). \quad (16)$$

III. ELEMENTARY EXCITATIONS IN THE POLARIZED BEC: DISPERSION EQUATION

We can analyze the linear dynamics of collective excitations in the polarized BEC using the QHD equations (4), (5), (11), (12) and (13)-(16). Let's assume the system is placed in an external electrical field $\mathbf{E}_0 = E_0 \mathbf{e}_z$. The values of concentration n_0 and polarization $\mathbf{P}_0 = \kappa \mathbf{E}_0$ for the system in an equilibrium state are constant and uniform and its velocity field $v^\alpha(\mathbf{r}, t)$, magnetic field $B^\alpha(\mathbf{r}, t)$ and tensor $R^{\alpha\beta}(\mathbf{r}, t)$ values are zero.

We consider the small perturbation of equilibrium state

$$n = n_0 + \delta n, \quad v^\alpha = 0 + \delta v^\alpha,$$

$$E^\alpha = E_0^\alpha + \delta E^\alpha, \quad B^\alpha = 0 + \delta B^\alpha,$$

$$P^\alpha = P_0^\alpha + \delta P^\alpha, \quad R^{\alpha\beta} = 0 + \delta R^{\alpha\beta}. \quad (17)$$

Substituting these relations into system of equations (4), (5), (11), (12) and (13)-(16) and neglecting nonlinear terms, we obtain a system of the linear homogeneous equations in partial derivatives with constant coefficients. Passing to the following representation for small perturbations δf

$$\delta f = f(\omega, \mathbf{k}) \exp(-i\omega t + i\mathbf{k}\mathbf{r})$$

yields the homogeneous system of algebraic equations. The electric field strength is assumed to have a nonzero value. Expressing all the quantities entering the system of equations in terms of the electric field, we come to the equation

$$\Lambda^{\alpha\beta}(\omega, \mathbf{k}) E^\beta(\omega, \mathbf{k}) = 0, \quad (18)$$

where

$$\Lambda^{\alpha\beta} = \left(\frac{\omega^2}{c^2} - k^2 \right) \delta^{\alpha\beta} + k^\alpha k^\beta - 4\pi \frac{\omega^2}{c^2} k^2 \frac{P_0^\alpha P_0^\beta \left(\frac{\sigma}{mn_0} + \frac{\frac{\omega k^2 \Upsilon}{2m^2 n_0}}{\omega^2 - \frac{\hbar^2 k^4}{4m^2} + \frac{\Upsilon n_0 k^2}{m} - \frac{\Upsilon_2 k^4 n_0}{m}} \right)}{\omega^2 - \frac{\hbar^2 k^4}{4m^2} + \frac{\Upsilon k^2 n_0}{2m}}. \quad (19)$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$.

We suppose that the external electric field parallel to the z-axis. Thereby, the equilibrium polarization \mathbf{P}_0 also parallel to the z-axis $\mathbf{P}_0 \parallel \mathbf{e}_z$. In this case tensor (19) become more simple

$$\hat{\Lambda}(\omega, \mathbf{k}) = \begin{pmatrix} \frac{\omega^2}{c^2} - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_x k_y & \frac{\omega^2}{c^2} - k_x^2 - k_z^2 & k_y k_z \\ k_x k_z & k_y k_z & \frac{\omega^2}{c^2} (1 + k^2 \beta(\omega)) - k_x^2 - k_y^2 \end{pmatrix}, \quad (20)$$

where

$$\beta(\omega) \equiv -4\pi \frac{\frac{P_0^2}{n_0} \left(\frac{\sigma}{m} + \frac{\omega k^2 \Upsilon}{2(m^2 \omega^2 - \hbar^2 k^4 / 4 + m \Upsilon n_0 k^2 - m \Upsilon_2 k^4 n_0)} \right)}{\omega^2 - \frac{\hbar^2 k^4}{4m^2} + \frac{\Upsilon k^2 n_0}{2m}}$$

Dispersion equation is $\det \hat{\Lambda} = 0$. Using formula (20) we have evident form of dispersion equation

$$(\omega^2 - k^2 c^2)^2 + \beta(\omega)(\omega^4 - (1 + \cos^2 \theta) \omega^2 k^2 c^2 + \cos^2 \theta k^4 c^4) = 0. \quad (21)$$

IV. DISPERSION DEPENDENCIES

In the low frequency limit ($\omega \ll kc$) from equation (21) we find

$$1 + \cos^2 \theta \beta(\omega) = 0, \quad (22)$$

at $\cos^2 \theta = 1$ we have

$$1 + \beta(\omega) = 0. \quad (23)$$

Equation (23) was obtained in Ref.s [13], [14], [15] and was in detail studied in Ref. [13].

If $P_0 = 0$ we find $\omega = kc$.

Equation (22) has following solution

$$\begin{aligned} \omega^2 = & \frac{1}{2m} \left(-\frac{3}{2} \Upsilon n_0 k^2 + \frac{\hbar^2 k^4}{2m} + \Upsilon_2 n_0 k^4 \right. \\ & + 4\pi\sigma \cos^2 \theta \frac{P_0^2 k^2}{n_0} \pm \left(\left(\frac{1}{2} \Upsilon n_0 k^2 - \Upsilon_2 n_0 k^4 \right. \right. \\ & \left. \left. + 4\pi\sigma \cos^2 \theta \frac{P_0^2 k^2}{n_0} \right)^2 - 8\pi \Upsilon k^4 \cos^2 \theta P_0^2 \right)^{1/2} \Bigg). \end{aligned} \quad (24)$$

In the case of wave propagation parallel to the equilibrium polarization $\cos \theta = 1$ solution we get solution obtained in Ref.s [13], [14], [15], and at great length considered in Ref. [13]. At increasing of angle θ from 0 to $\pi/2$ contribution of equilibrium polarization become smaller and vanish at $\theta = \pi/2$. Obtaining of an anisotropic dispersion dependence emerges because of consideration of the whole set of the Maxwell's equations (13)-(16). It gives us possibility to account transverse electric field in the wave, along with the longitudinal field, the last one was considered in our previous papers [13], [14], [15].

Under the condition $\theta = 0$ or $\theta = \pi$ equation (21) assumes the form

$$(\omega^2 - k^2 c^2)^2 (1 + \beta(\omega)) = 0. \quad (25)$$

From this equation we find usual the dispersion dependence of the light $\omega = kc$, and equation $1 + \beta(\omega) = 0$ describes the two matter waves, which appear instead of the Bogoliubov's mode in the electrically polarized BEC, see Ref.s [13], [14], [15].

In the opposite case, when $\theta = \pi/2$ equation (21) simplifies to

$$(\omega^2 - k^2 c^2)(\omega^2 - k^2 c^2 + \beta(\omega)\omega^2) = 0. \quad (26)$$

In this case we have two independent equation. One of them $\omega^2 - k^2 c^2 = 0$ describes changeless dispersion of the light. The second equation

$$\omega^2 (1 + \beta(\omega)) - k^2 c^2 = 0 \quad (27)$$

is the equation of the third degree of ω^2 . So, it contains three wave branches. We can admit that it contains both modified by the BEC dispersion of the light and the two matter wave. This equation has no low frequencies limitation as (22), thus we have here high frequencies "tail" in the dispersion of the matter waves.

Let's consider influence of the BEC on the light as a small correction. Therefore we put $\omega = kc$ in $\beta(\omega)$ of equation (27), and accounting that $kc \gg \hbar k^2/m$, $\sqrt{\Upsilon n_0/mk}$, $\sqrt{\Upsilon_2 n_0/mk^2}$, in the result we find

$$\omega \simeq kc / \sqrt{1 + \beta(kc)}, \quad (28)$$

where $\beta(kc) = -2\pi \Upsilon P_0^2 / (m^2 n_0 k c^3)$. Expanding formula (28) in a series by $\beta(kc)$ we get

$$\omega = kc \left(1 + \frac{\pi \Upsilon P_0^2}{m^2 n_0 k c^3} \right).$$

We can see that the polarized BEC caused splitting of the light on two waves at light propagation perpendicular to the direction of the equilibrium polarization. Magnitude of splitting M_s is equal to

$$M_s = \frac{\pi \Upsilon P_0^2}{m^2 n_0 c^2}. \quad (29)$$

V. CONCLUSION

We have considered transverse waves in the electrically polarized BEC. To do it we have used whole set of the Maxwell's equation for description of electromagnetic field caused by the dynamic of electric dipoles of the medium. In comparison to the case of longitudinal waves considered in our previous papers we have found two advantages.

The first of them is the possibility of considering of light propagating through the electrically polarized BEC and studying of it's properties. Thus, we have considered dispersion of the light propagating through the medium and we have found the splitting of the light on two waves in the case when the light propagate at the angle to the direction of the equilibrium polarization. We have calculated the magnitude of frequencies splitting for the case of the light propagation perpendicular to the direction of equilibrium polarization. There is no splitting in the case of the light propagation along the direction of the equilibrium polarization, there is also no contribution of the medium in the light dispersion dependence in this case.

The second advantage of the considered approximation is the appearing of the anisotropy in the dispersion dependencies of the matter waves. As in the case of longitudinal waves we have got two waves which exist instead of the Bogoliubov's mode existing in the unpolarized BEC. However, we did not find, in our previous papers, dependence of the frequencies of the collective excitation on the direction of wave propagation for the longitudinal waves, which might be expected due to the anisotropy of the Hamiltonian of dipole-dipole interaction. We have found analytical solution for the plane matter waves propagating in the three dimensional BEC and shown the influence of the anisotropy on the value of the frequencies.

In the result, we have developed generalization of the quantum hydrodynamic equations for the electrically polarized BEC including transverse wave propagation. Using it we have shown that in the electrically polarized BEC situated in the external electric field there are four wave, two of them are high frequencies waves, and them associated with the light. Two more wave are the anisotropic matter waves.

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